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LETTER TO THE EDITOR

Limitations on universality in the continuous-spin Ising model†

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Abstract. We give by explicit counterexample numerical evidence that critical exponent universality fails for the two-dimensional, continuous-spin, Ising model. This result is contrary to what has previously been assumed.

To date, the most quantitative implementation of the renormalisation group theory of critical phenomena (see, for example, Ma 1978, and particularly the references therein) is the approach (Brezin *et al* 1976) based on the renormalised perturbation theory expansion of the Callan–Symanzik equations for $\lambda : \phi^4$: Euclidean, Boson field theory. The idea that this approach should describe the spin-$\frac{1}{2}$ Ising model depends on a universality hypothesis (Baker and Kincaid 1979, 1981, and Parisi 1980). In the context of the continuous-spin Ising model, this universality hypothesis takes the mathematical form that certain double limits are independent of their order.

Baker (1983) has investigated some of the consequences of this universality hypothesis for the continuous-spin Ising model. Specifically, he considered the partition function defined by

$$Z = M^{-1} \int_{-\infty}^{\infty} \cdots \int \prod_{i=1}^{N} \mathrm{d}s_i \exp\left[\sum_i \left(K \sum_{\{\delta\}} s_i s_{i+\delta} - \tilde{g}_0 s_i^4 - \tilde{A} s_i^2 + H s_i\right)\right], \quad (1)$$

where M is a formal normalisation constant such that $Z(H = K = 0) = 1$, N is the number of lattice sites, $\{\delta\}$ is one half the set of nearest-neighbour lattice vectors, and $K = J/kT$ with J the exchange integral, k Boltzmann's constant and T the absolute temperature. A normalisation of the spin variable is imposed by the requirement that

$$1 = \langle s^2 \rangle = \int_{-\infty}^{\infty} s^2 \exp(-\tilde{g}_0 s^4 - \tilde{A} s^2)\, \mathrm{d}s \bigg/ \int_{-\infty}^{\infty} \exp(-\tilde{g}_0 s^4 - \tilde{A} s^2)\, \mathrm{d}s. \quad (2)$$

Equation (2) defines $\tilde{A}$ as a function of $\tilde{g}_0$ such that $\tilde{A}$ is analytic in the region $0 < \tilde{g}_0 < \infty$ and ranges from $\tilde{A}(0) = \frac{1}{2}$ to $\lim_{\tilde{g}_0 \to \infty} \tilde{A}/\tilde{g}_0 = -2$. The well known Gaussian model corresponds to $\tilde{g}_0 = 0$ and the Ising model to $\tilde{g}_0 \to \infty$.

From the universality assumption that ν is independent of $\tilde{g}_0$ and from the representation for the spin–spin correlation length (second moment definition) near the

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critical point, $T > T_c$,

$$\xi \cong D_+(\tilde{g}_0)[1 - K/K_c(\tilde{g}_0)]^{-\nu}, \tag{3}$$

Baker (1983) found for spatial dimensionality, $d \geqslant 2$, the amplitude formula

$$[D_+(\tilde{g}_0)]^{1/\nu} = \theta(K_c(\tilde{g}_0)/\tilde{A}(\tilde{g}_0))\tilde{g}_0^{[(2-1/\nu)/(4-d)]}. \tag{4}$$

Here θ is a constant independent of $\tilde{g}_0$. Certain aspects of the formula are understandable from what is now known or believed about the continuous-spin Ising model. For $d = 2$ or 3, equation (4) shows that $D_+(\tilde{g}_0) \to 0$ as $\tilde{g}_0 \to 0$; this behaviour, for example, indicates the differences in ν between the Gaussian model at $\tilde{g}_0 \equiv 0$ and models for $\tilde{g}_0 > 0$. The models with $\tilde{g}_0 > 0$ should reflect the strong coupling limit of the field theory with the corresponding 'universal' value for ν since $\tilde{g}_0 \propto g_0/\xi^{4-s}$ (Baker and Kincaid 1981). Conversely, any finite, bare, boson field theory, coupling constant g_0 implies $\tilde{g}_0 = 0$ at the critical point. For $d = 4$, equation (4) is not well defined since $4 - d$ is in the denominator of the exponent. For $d > 4$, ν is thought to be $\frac{1}{2}$ implying that equation (4) does not vanish as $\tilde{g}_0 \to 0$; this result is in agreement with Aisenman's (1981) result that the critical index for the magnetic susceptibility agrees with the Gaussian model result for $d > 4$. These results are all consistent with the conventional picture. However, equation (4) suggests further details that imply the conventional picture does not extend from small values of $\tilde{g}_0$ all the way to $\tilde{g}_0 = \infty$. In particular, $\tilde{A}(g_b) = 0$ for $g_b = [\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})]^2 \cong 0.114\,236\,6452$ which forces D_+ to diverge as $\tilde{g}_0 \to g_b$ to maintain the universality assumption just mentioned. This behaviour of the amplitude can be interpreted as characteristic of a cross over from one value to ν to another.

Further analysis of equation (4) shows that there are five different cases; i.e., (i) $\tilde{g}_0 = 0$, (ii) $0 < \tilde{g} < g_b$, (iii) $\tilde{g}_0 = g_b$, (iv) $g_b < \tilde{g}_0 < \infty$, and (v) $\tilde{g}_0 = \infty$. That case (v) differs from case (iv) is found by combining equation (4) with conventional values of ν and $K_c(\tilde{g}_0) < \infty$ for $d \geqslant 2$ to imply that $D_+ \to 0$ as $\tilde{g}_0 \to \infty$. Finally, we note that cases (i) and (ii) have a single-peaked spin distribution with a quadratic top while case (iv) has a quadratic minimum at $s = 0$ with two rounded peaks.

As a consequence of the above analysis, we have elected to investigate case (iii), which we call the border case. Using the computer files of Baker and Kincaid (1981), we have computed the high temperature expansion for the magnetic susceptibility, χ, the correlation length squared, ξ^2, and $(\partial^2\chi/\partial H^2)/\chi$, where H is the magnetic field, through the tenth order. These series were computed for the linear, plane square, triangular, simple cubic, body-centred cubic, face-centred cubic, hyper-simple cubic, and hyper-body-centred cubic lattices. We have analysed all these series and found a number of results which may be of interest. However, for the present we will concentrate on the magnetic susceptibility in two dimensions. The coefficients we obtained are listed in table 1. We have analysed these coefficients in several ways. First, by forming Padé approximants (see, for example, Baker and Graves-Morris 1981) to the logarithmic derivative, we obtain the estimates for the behaviour of the magnetic susceptibility near the critical point $\chi \propto (K_c - K)^{-\gamma}$

$$\begin{array}{lll} \text{plane square} & K_c = 0.329 \pm 0.002, & \gamma = 1.92 \pm 0.06, \\ \text{triangular} & K_c = 0.2130 \pm 0.0005, & \gamma = 1.92 \pm 0.03 \end{array} \tag{5}$$

where the apparent errors are estimated by the standard Hunter–Baker (Hunter and Baker 1973) prescription. In addition we have used a ratio based method due to Zinn–Justin (1979) which does not explicitly involve the coefficient subscript n and is a

Table 1. Series coefficients.

n	plane square lattice	triangular lattice
0	1	1
1	4	6
2	14.376 879 230 452 953 277 6735	33.565 318 845 679 429 916 5103
3	51.454 120 682 379 459 080 4787	182.573 094 097 645 768 119 780
4	176.601 080 981 470 582 306 466	971.950 597 865 016 727 198 828
5	604.310 878 998 466 269 971 823	5094.953 177 182 493 321 656 90
6	2021.518 731 265 287 358 789 24	26 390.663 148 511 955 195 8484
7	6745.483 241 921 079 131 303 86	135 396.839 691 927 325 286 174
8	22 192.709 557 176 507 411 6227	689 235.301 903 841 609 817 108
9	72 853.820 700 277 499 505 3795	3485 628.814 904 977 215 187 38
10	236 889.741 875 463 268 566 817	17 529 844.571 519 168 341 6769

second-order method. The last four estimates, for each lattice are

$$\begin{array}{llllll} \text{plane square} & K_c & 0.327\,640, & 0.327\,607, & 0.328\,788, & 0.328\,855, \\ & \gamma & 1.8264, & 1.8096, & 1.8849, & 1.8791, \\ \text{triangular} & K_c & 0.212\,504, & 0.212\,505, & 0.212\,904, & 0.212\,903 \\ & \gamma & 1.8679, & 1.8680, & 1.9081, & 1.9080. \end{array} \tag{6}$$

We do not have a method to estimate reliably the error for this method; however, these limits appear to be consistent, within error, with those quoted in (5). Note that the results for the plane square are based on Zinn–Justin's even–odd adaption because of the anti-ferromagnetic singularity present in that case.

In the case where there could be a confluent singularity, we know that the results of the direct analysis may be biased. For this reason, we have performed a Baker–Hunter (Baker and Hunter 1973) confluent singularity analysis. It is geared to the idea that near the critical point

$$\chi \cong \sum_i A_i(1-K/K_c)^{-\gamma_i}. \tag{7}$$

The method requires a hypothesis concerning K_c and yields estimates for γ_i and A_i. We have moved the anti-ferromagnetic critical point to the vicinity of infinity by a Gaunt–Sykes (1979) transformation of the plane square lattice series before performing the confluent singularity analysis. We have scanned the region of K_c given by (5) and found that, as one might guess from the work on the other series of Adler *et al* (1982a, b), there is a significant narrowing of the scatter of the various central Padé approximants at an appropriately selected value of the critical temperature. We find best results at

$$\begin{array}{llll} \text{plane square} & K_c=0.3300, & \gamma_1=1.996\pm0.02, & \gamma_2=1.042\pm0.02, \\ \text{triangular} & K_c=0.2134, & \gamma_1=2.001\pm0.01, & \gamma_2=1.063\pm0.01. \end{array} \tag{8}$$

The ranges quoted represent the variation in the estimates with Padé approximant selected at fixed K_c and are not to be interpreted as error estimates. The region in the Padé table of small, fluctuation is larger for the triangular than for the square lattice. The results (8) give for the correction to scaling index $\Delta_1=\gamma_1-\gamma_2\cong0.938$

which, while near $\Delta_1 = 1$ is nevertheless a confluent singularity and responsible, in our opinion, for the differences between (5) and (6), and (8). In this case, contrary to the three-dimensional Ising model, the amplitude of the subdominate singularity is substantial. We find that

$$\text{plane square}\quad K_c = 0.3300,\quad \zeta \cong 0.061\,70(1-K/K_c)^{-2.00} + 0.4825(1-K/K_c)^{-1.05},$$

$$\text{triangular}\quad K_c = 0.2134,\quad \chi \cong 0.2278(1-K/K_c)^{-2.00} + 0.7707(1-K/K_c)^{-1.06},$$

well represent the behaviour of χ near the critical point.

On the basis of these analyses we conclude that the index γ for the border model should be in the range $\gamma_b = 1.89$ to 2.02, and very likely close to $\gamma_b = 2.00$. Since it is known (McCoy and Wu 1973) for the two-dimensional Ising model ($\tilde{g}_0 = \infty$) that $\gamma_I = 1.75$, we conclude that $\gamma_b \neq \gamma_I$, and thus critical exponent universality does not hold for the continuous-spin Ising model ($0 < \tilde{g}_0 \leq \infty$). Of course, in the application of field-theoretic methods to the calculation of Ising model critical indices, universality was assumed. A note is made that direct estimates by that method (Baker *et al* 1978) lead to $\gamma = 1.72 \pm 0.2$, $\Delta_1 = 1.4 \pm 0.8$ (there is a factor of two error in (4.16) of this reference) are sufficiently uncertain as not to shed any light on this question. We further note the results of Klauder (1982) in four-dimensions which suggest possible critical exponent dependence on the single-spin distribution function.

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